B.Sc. Part-I(H) Paper-I Abst Algebra 24-07-2020

Q.1. If a and b are any two elements of a group (G) then $o(a) = o(b^{-1}ab)$

Proof:→ Let $o(a) = m$, hence m is the least positive integer, such that $a^m = e$

Now, $(b^{-1}ab)^2 = (b^{-1}ab)(b^{-1}ab)$
$= b^{-1}a(bb^{-1})ab$ [By associativity]
$= b^{-1}aeab$ [$\because bb^{-1} = e$]
$= b^{-1}a^2b$ [$\because ae = a$]

Similarly, $(b^{-1}ab)^3 = b^{-1}a^3b$ ...... and so on.

$(b^{-1}ab)^m = (b^{-1}ab)(b^{-1}ab)\ldots$ to m factors
$= b^{-1}abb^{-1}ab \ldots b^{-1}ab$ [By associativity]
$= b^{-1}a(bb^{-1})a(bb^{-1})\ldots(bb^{-1})ab$ [By associativity]
$= b^{-1}a^mb = b^{-1}eb = b^{-1}b = e$ [$\because a^m = e$]

Thus, we have $(b^{-1}ab)^m = b^{-1}a^mb = e$

Now, since, m is the least positive integer such that $a^m = e$, it follows that m is the least positive integer such that
$(b^{-1}ab)^m = e$

Hence, $o(b^{-1}ab) = m$ proved.

Q.2. If a is an element of order n and p is prime to n, then a^p is also of order n.

proof: Let r be the order of a^p.

Now, $o(a) = n \Rightarrow a^n = e$
$\Rightarrow (a^n)^p = e^p = e \Rightarrow (a^p)^n = e \Rightarrow o(a^p) \leq n \Rightarrow r \leq n$ —(1)

Now, since p, n are relatively prime, there exist

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integers x and y such that $px + ny = 1$

$$a = a^1 = a^{px+ny} = a^{px} \cdot a^{ny} = a^{px}(a^n)^y = a^{px} e^y$$
$$= a^{px} e^y = a^{px} \cdot e = a^{px} = (a^p)^x$$

Also, $a^r = [(a^p)^x]^r = (a^p)^{rx} = [(a^p)^r]^x = e^x = e$

$[\because \; o(a^p) = r \Rightarrow (a^p)^r = e]$

$\therefore \; o(a) \leq r \Rightarrow n \leq r$ —— (2)

From (1) and (2), we conclude that
$n = r$

Proved.

Q.3. The order of any integral power of an element of a group is a divisor of the order of that element

Proof: Let a be any element of a group G with $o(a) = n$

$\Rightarrow a^n = e$

Let a^r be any integral power of a such that $o(a^r) = m$

Then, $(a^r)^n = a^{rn} = (a^n)^r = e^r = e$

Thus, $(a^r)^n = e \Rightarrow o(a^r)$ divides n

$\Rightarrow o(a^r)$ is a divisor of $o(a)$.

Proved.